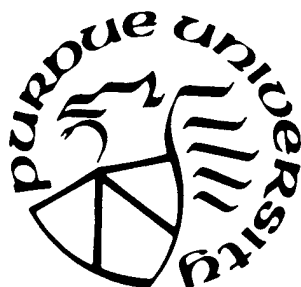




# **PURDUE UNIVERSITY**



**DEPARTMENT OF STATISTICS**

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ON SELECTION PROCEDURES FOR EXPONENTIAL FAMILY  
DISTRIBUTIONS BASED ON TYPE-I CENSORED DATA\*

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# On Selection Procedures For Exponential Family Distributions Based On Type-I Censored Data\*

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## Abstract

We investigate the problem of selecting the best population from exponential family distributions based on type-I censored data. A Bayes rule is derived and a monotone property of the Bayes selection rule is obtained. Following that property, we propose an early selection rule. Through this early selection rule, one can terminate the experiment on a few populations early and possibly make the final decision before the censoring time. An example is provided in the final part to illustrate the use of the early selection rule for Weibull populations.

AMS Classification: 62F07, 62L10.

Keywords: Type-I censored data, best population, Bayes selection rule, early selection rule.

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## § 1 Introduction

Consider designing and analyzing an experiment for comparing  $k$  populations  $\pi_1, \pi_2, \dots, \pi_k$ . Suppose that  $m$  items are taken from each population and observations can be obtained from those items in time order, as for example, in a life-testing experiment. It is often desirable to terminate the test from a population as soon as there is enough statistical evidence that it is not the best population, and then this population is eliminated from further consideration.

Assume that the random observations from population  $\pi_i$  have a density function  $f(x|\theta_i)$  of the form

$$f(x|\theta) = c(\theta)\exp\{\beta(\theta)Q(x)\}h(x), \quad x \in \mathcal{X}, \quad (1.1)$$

where  $\mathcal{X}$  is the support of  $f(x|\theta)$  and  $h(x) > 0$  for  $x \in \mathcal{X}$ . Let  $\Omega$  be the parameter space for each  $\theta_i$ .

Let  $\theta_{[1]} \leq \theta_{[2]} \leq \dots \leq \theta_{[k]}$  denote the ordered values of the parameters  $\theta_1, \theta_2, \dots, \theta_k$ . It is assumed that the exact pairing between the ordered and the unordered parameters is unknown. The population associated with the largest value  $\theta_{[k]}$  is considered as the best population. The readers are referred to Gupta and Panchapakesan (1979) for a comprehensive understanding of selection and ranking procedures.

The function  $f(x|\theta)$  is assumed to have (nondecreasing) monotone likelihood ratio with respect to  $x$ . This assumption is equivalent to

$$(\beta(\theta_2) - \beta(\theta_1))(Q(x_2) - Q(x_1)) \geq 0 \quad (1.2)$$

for any  $\theta_2 \geq \theta_1$ ,  $\theta_1, \theta_2 \in \Omega$ ,  $x_2 \geq x_1$ ,  $x_1, x_2 \in \mathcal{X}$ . Without loss of generality, we assume that  $Q(x)$  is a nondecreasing function of  $x$  and  $\beta(\theta)$  is a nondecreasing function of  $\theta$ .

Many exponential family distributions, such as *Chi-square*, *Exponential*, *Gamma* ( $\alpha, \beta$ ) with one of the two parameters known, *Log-normal* ( $\mu, \sigma^2$ ) with  $\sigma$  known, *Weibull* ( $\gamma, \beta$ ) with one of the two parameters known, have property (1.2). So our results here can be applied to them.

In an application situation of industrial life-testing experiment,  $m$  items from each of the  $k$  population  $\pi_1, \dots, \pi_k$  are independently put on test at the outset and are not replaced on failure. Due to the time restriction, the experiment terminates at a pre-specified time  $T$ . The failure time of an item is observable only if it fails before time  $T$ . Otherwise the item is said to be censored at time  $T$ . This type of time censoring is known as the type-I censoring. The type-I censoring scheme has received much attention in the statistical literature, see Spurrier and Wei (1980), and others. The ranking and selection procedures based on censored data for the exponential distribution have been considered, for example, in Berger and Kim (1985), Gupta and Liang (1993), and Huang and Huang (1980).

In this paper, we derive a Bayes selection rule for exponential family distributions based on type-I censored data. A monotone property of this rule is discussed and an

early selection rule is proposed. Through this early selection rule, one can terminate the experiment on a few populations early and possibly make the final decision before the given time  $T$ . The approach used here is similar to that of Gupta and Liang(1993).

## §2 A Bayes Selection Rule

Let  $T$  be the censoring time. Let  $X_{ij}, 1 \leq j \leq m$  be the type-I censored data of the  $m$  items taken from population  $\pi_i$ . We only observe  $\min(X_{ij}, T)$ . Let  $N_i = \sum_{j=1}^m I_{[X_{ij} < T]}$  be the number of uncensored data up to time  $T$ .

Let  $Y_{i1} \leq Y_{i2} \leq \dots \leq Y_{iN_i}$  be the ordered values of the  $N_i$  observed data given  $N_i$ . Then  $(Y_{i1}, Y_{i2}, \dots, Y_{iN_i}, N_i)$  have a joint probability density

$$\begin{aligned} f(y_i, n_i | \theta_i) &= \frac{m!}{(m - n_i)!} c^{n_i}(\theta_i) e^{\beta(\theta_i) \sum_{j=1}^{n_i} Q(y_{ij})} P_{\theta_i}(X \geq T)^{(m - n_i)} \prod_{j=1}^{n_i} h(y_{ij}) \\ &= \frac{m!}{(m - n_i)!} c^{n_i}(\theta_i) e^{\beta(\theta_i)[y_i - (m - n_i)Q(T)]} P_{\theta_i}(X \geq T)^{(m - n_i)} \prod_{j=1}^{n_i} h(y_{ij}), \end{aligned} \quad (2.1)$$

where  $y_i = (y_{i1}, y_{i2}, \dots, y_{in})$ ,  $0 \leq n \leq m$ ,  $y_{i1} \leq y_{i2} \leq \dots \leq y_{in} < T$  and

$$y_i = \sum_{j=1}^{n_i} Q(y_{ij}) + (m - n_i)Q(T). \quad (2.2)$$

Let  $\mathcal{N}$  be the sample space generated by  $\tilde{N} = (n_1, n_2, \dots, n_k)$  and conditioned on  $\tilde{N} = \tilde{n} = (n_1, n_2, \dots, n_k)$ , let  $\mathcal{Y}_{\tilde{n}}$  be the sample space generated by  $\tilde{Y} = (y_1, y_2, \dots, y_k)$ .

Let  $\tilde{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$  and  $\tilde{\Omega} = \{\tilde{\theta} | \theta_i \in \Omega, 1 \leq i \leq k\}$  be the parameter space. Let  $\mathcal{A}$  be the action space. Action  $i$  corresponds to the selection of population  $\pi_i$  as the best population. For a given  $\theta \in \Omega$  and an action  $i$ , the associated loss function is defined by

$$L^*(\theta, i) = L(\theta_{[k]} - \theta_i) \quad (2.3)$$

where  $L(x)$  is a nonnegative and nondecreasing function of  $x$ ,  $x \geq 0$ , such that  $L(0) = 0$ .

Let  $g(\tilde{\theta}) = \prod_{j=1}^k g_j(\theta_j)$  be the prior density over the parameter space  $\tilde{\Omega}$ . It is assumed that  $\int_{\tilde{\Omega}} L(\theta_{[k]}) g(\tilde{\theta}) d\tilde{\theta} < \infty$ .

A selection rule  $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_k)$  is defined to be a measurable mapping from the sample space  $(\mathcal{N}, (\mathcal{Y}_{\tilde{n}})_{\tilde{n} \in \mathcal{N}})$  to  $[0, 1]^k$  such that  $0 \leq \delta_i(\tilde{y}, \tilde{n}) \leq 1$  and  $\sum_{j=1}^k \delta_j(\tilde{y}, \tilde{n}) = 1$  for all  $\tilde{y} \in \mathcal{Y}_{\tilde{n}}, \tilde{n} \in \mathcal{N}$ . The value of  $\delta_i(\tilde{y}, \tilde{n})$  is the probability of selecting population  $\pi_i$  as the best population based on the observation  $(\tilde{y}, \tilde{n})$ .

Let  $R(\tilde{\delta}, g)$  denote the Bayes risk associated with the selection rule  $\tilde{\delta}$ . Then by Fubini's Theorem we have

$$R(\tilde{\delta}, g) = \sum_{\tilde{n} \in \mathcal{N}} \int_{\mathcal{Y}_{\tilde{n}}} \sum_{i=1}^k \delta_i(\tilde{y}, \tilde{n}) \int_{\Omega} L(\theta_{[k]} - \theta_i) f(\tilde{y}, \tilde{n} | \tilde{\theta}) g(\tilde{\theta}) d\tilde{\theta} d\tilde{y} \quad (2.4)$$

where  $f(\tilde{y}, \tilde{n}|\tilde{\theta}) = \prod_{i=1}^k f(y_i, n_i|\theta_i)$ . Now let

$$f_i(y_i, n_i) = \int_{\Omega} f(y_i, n_i|\theta_i) g_i(\theta_i) d\theta_i, \quad f(\tilde{y}, \tilde{n}) = \prod_{i=1}^k f_i(y_i, n_i),$$

$$g_i(\theta_i|y_i, n_i) = \frac{f(y_i, n_i|\theta_i) g_i(\theta_i)}{f_i(y_i, n_i)} \quad \text{and} \quad g(\tilde{\theta}|\tilde{y}, \tilde{n}) = \prod_{i=1}^k g_i(\theta_i|y_i, n_i).$$

Then (2.4) becomes

$$R(\tilde{\delta}, g) = \sum_{\tilde{n} \in \mathcal{N}} \int_{\mathcal{Y}_{\tilde{n}}} \sum_{i=1}^k \delta_i(\tilde{y}, \tilde{n}) \int_{\tilde{\Omega}} L(\theta_{[k]} - \theta_i) g(\tilde{\theta}|\tilde{y}, \tilde{n}) d\tilde{\theta} f(\tilde{y}, \tilde{n}) d\tilde{y}.$$

For each  $(\tilde{y}, \tilde{n})$ , define

$$\Delta_i(\tilde{y}, \tilde{n}) = \int_{\tilde{\Omega}} L(\theta_{[k]} - \theta_i) g(\tilde{\theta}|\tilde{y}, \tilde{n}) d\tilde{\theta}, \quad i = 1, 2, \dots, k, \quad (2.5)$$

and let

$$A(\tilde{y}, \tilde{n}) = \{i | \Delta_i(\tilde{y}, \tilde{n}) = \min_{1 \leq j \leq k} \Delta_j(\tilde{y}, \tilde{n})\}. \quad (2.6)$$

Then a uniformly randomized Bayes rule is  $\tilde{\delta}_G = (\delta_{G1}, \dots, \delta_{Gk})$ , where

$$\delta_{Gi} = \begin{cases} |A(\tilde{y}, \tilde{n})|^{-1} & \text{if } i \in A(\tilde{y}, \tilde{n}) \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

### §3 A Monotonicity Property of $\tilde{\delta}_G$

For each fixed  $(y_i, n_i)$ ,  $g_i(\theta_i|y_i, n_i) = 0$  if and only if  $g_i(\theta_i) = 0$ . Then  $g_i(\theta_i|y_i^*, n_i^*)$  and  $g(\theta_i|y_i, n_i)$  have the common support. Let  $D_i$  be their common support. Consider the likelihood ratio defined on  $D_i$ , by

$$r_i(\theta_i|y_i^*, n_i^*, y_i, n_i) = \frac{g_i(\theta_i|y_i^*, n_i^*)}{g_i(\theta_i|y_i, n_i)}. \quad (3.1)$$

A simple calculation shows that for some nonnegative function  $W$

$$\begin{aligned} & r_i(\theta_i|y_i^*, n_i^*, y_i, n_i) \\ &= W(y_i^*, n_i^*, y_i, n_i) c^{n_i^* - n_i}(\theta_i) e^{\beta(\theta_i)[(y_i^* - y_i) + (n_i^* - n_i)Q(T)]} \{P_{\theta_i}(X \geq T)\}^{(n_i - n_i^*)} \\ &= W(y_i^*, n_i^*, y_i, n_i) e^{\beta(\theta_i)(y_i^* - y_i)} \left\{ \int_T^\infty \exp\{\beta(\theta_i)[Q(x) - Q(T)]\} h(x) dx \right\}^{(n_i - n_i^*)}, \end{aligned}$$

from which we get the following lemma.

**Lemma 3.1** *Let  $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$  be defined by (3.1). Then*

(a) *for  $n_i^* = n_i, y_i^* > y_i$ ,  $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$  is a nondecreasing function of  $\theta_i$  in  $D_i$  and*

(b) *for  $y_i^* = y_i, n_i^* > n_i$ ,  $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$  is a nonincreasing function of  $\theta_i$  in  $D_i$ .*

The following lemma is used in the proof of lemma 3.3.

**Lemma 3.2** *If  $g(\theta)$  and  $h(\theta)$  are probability density functions such that  $g(\theta)/h(\theta)$  is nondecreasing function of  $\theta$  in  $\Omega$ , then for any nonincreasing function  $f(\theta)$  of  $\theta$  in  $\Omega$ ,*

$$\int_{\Omega} f(\theta)h(\theta)d\theta \geq \int_{\Omega} f(\theta)g(\theta)d\theta.$$

**Lemma 3.3** *Let  $\Delta_i(\tilde{y}, \tilde{n})$  be defined in (2.5). For each  $i(1 \leq i \leq k)$ ,  $\Delta_i(\tilde{y}, \tilde{n})$  is nonincreasing in  $y_i$  and also in  $n_j, j \neq i$  when all the other variables are kept fixed, and nondecreasing in  $n_i$  and also in  $y_j, j \neq i$ , when all the other variables are kept fixed.*

**Proof.** We only prove that  $\Delta_i(\tilde{y}, \tilde{n})$  is nonincreasing in  $y_i$  when all the other variables are kept fixed. The other parts can be proved in a similar way.

Define

$$\begin{aligned} \tilde{\theta}^i &= (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_k) \\ \tilde{\Omega}^i &= \{\tilde{\theta}^i : \theta_j \in \Omega, j = 1, 2, \dots, k, j \neq i\} \\ \tilde{y} &= (y_1, \dots, y_k) \\ \tilde{y}^* &= (y_1, \dots, y_{i-1}, y_i^*, y_{i+1}, \dots, y_k). \end{aligned}$$

Then

$$\Delta_i(\tilde{y}, \tilde{n}) = \int_{\tilde{\Omega}^i} \left[ \int_{\Omega} L(\theta_{[k]} - \theta_i) g_i(\theta_i|y_i, n_i) d\theta_i \right] \prod_{j \neq i} g_j(\theta_j|y_j, n_j) d\tilde{\theta}^i.$$

Since for each fixed  $\tilde{\theta}^i$  and  $\tilde{n}$ ,  $L(\theta_{[k]} - \theta_i)$  is nonincreasing in  $\theta_i$  and by Lemma 3.1,  $r_i(\theta_i|y_i^*, n_i^*, y_i, n_i)$  is a nondecreasing function of  $\theta_i$  for  $y_i^* > y_i$ . So Lemma 3.2 implies that

$$\int_{\Omega} L(\theta_{[k]} - \theta_i) g_i(\theta_i|y_i, n_i) d\theta_i \geq \int_{\Omega} L(\theta_{[k]} - \theta_i) g_i(\theta_i|y_i^*, n_i) d\theta_i$$

and hence  $\Delta_i(\tilde{y}, \tilde{n}) \geq \Delta_i(\tilde{y}^*, \tilde{n})$ .

Now, from Lemma 3.3, we obtain a monotone property for  $\tilde{\delta}_G$  in the following theorem.

**Theorem 3.4** *For each  $i = 1, 2, \dots, k$ ,  $\delta_{Gi}(\tilde{y}, \tilde{n})$  is nondecreasing in  $y_i$  and nonincreasing in  $n_i$ , when all the other variables are kept fixed.*

**Proof.** We only prove that  $\delta_{Gi}(\tilde{y}, \tilde{n})$  is nondecreasing in  $y_i$  when all other variables are kept fixed. The monotone property of  $\tilde{\delta}_G$  in  $n_i$  can be proved in a similar way.

Use the notation in Lemma 3.3. Assume  $y_i^* > y_i$ .

If  $i \notin A(\tilde{y}, \tilde{n})$ , then  $\delta_{Gi}(\tilde{y}, \tilde{n}) = 0$ . Since  $\delta_{Gi}(\tilde{y}^*, \tilde{n})$  is nonnegative,  $\delta_{Gi}(\tilde{y}^*, \tilde{n}) \geq \delta_{Gi}(\tilde{y}, \tilde{n})$ .

If  $i \in A(\tilde{y}, \tilde{n})$ , then  $\Delta_i(\tilde{y}, \tilde{n}) \leq \min_{j \neq i} \Delta_j(\tilde{y}, \tilde{n})$ . Using Lemma 3.3,

$$\Delta_i(\tilde{y}^*, \tilde{n}) \leq \Delta_i(\tilde{y}, \tilde{n}) \leq \min_{j \neq i} \Delta_j(\tilde{y}, \tilde{n}) \leq \min_{j \neq i} \Delta_j(\tilde{y}^*, \tilde{n}).$$

And hence  $i \in A(\tilde{y}^*, \tilde{n})$ .

To get  $\delta_{Gi}(\tilde{y}^*, \tilde{n}) \geq \delta_{Gi}(\tilde{y}, \tilde{n})$ , we still need to show

$$A(\tilde{y}^*, \tilde{n}) \subset A(\tilde{y}, \tilde{n}). \quad (3.2)$$

For each  $h \in A(\tilde{y}^*, \tilde{n})$ ,  $\Delta_h(\tilde{y}^*, \tilde{n}) = \Delta_i(\tilde{y}^*, \tilde{n})$ . Using Lemma 3.3,

$$\Delta_h(\tilde{y}, \tilde{n}) \leq \Delta_h(\tilde{y}^*, \tilde{n}) = \Delta_i(\tilde{y}^*, \tilde{n}) \leq \Delta_i(\tilde{y}, \tilde{n}) = \min_{1 \leq j \leq k} \Delta_j(\tilde{y}, \tilde{n})$$

and hence  $h \in A(\tilde{y}, \tilde{n})$ . So (3.2) is proved and  $\delta_{Gi}(\tilde{y}^*, \tilde{n}) \geq \delta_{Gi}(\tilde{y}, \tilde{n})$ .

## §4 An Early Selection Rule

In this section, we consider the following linear loss function:  $L(\theta_{[k]} - \theta_i) = \theta_{[k]} - \theta_i$ , the difference between the parameters of the best and the selected populations. Thus the set  $A(\tilde{y}, \tilde{n})$  given by (2.6) turns out to be:

$$A(\tilde{y}, \tilde{n}) = \{i \mid \int \theta_i g_i(\theta_i | y_i, n_i) d\theta_i = \max_{1 \leq j \leq k} \int \theta_j g_j(\theta_j | y_j, n_j) d\theta_j\}. \quad (4.1)$$

Similar to the proof of Lemma 3.3, we can prove the following result.

**Lemma 4.1** *For each fixed  $i$ ,  $E[\theta_i | y_i, n_i]$  is increasing in  $y_i$  and decreasing in  $n_i$ .*

Now, we will use Lemma 4.1 to derive an early selection rule.

At time  $t$ ,  $0 < t < T$ , let  $N_i(t)$  denote the number of uncensored data from population  $\pi_i$  upto  $t$ . That is,  $N_i(t) = \#\{X_{ij} : 1 \leq j \leq m, X_{ij} \leq t\}$ . Also, let  $Y_{i1} \leq Y_{i2} \leq \dots \leq Y_{iN_i(t)}$  denote observed uncensored data given  $N_i(t)$ . At time  $t$ , we can make early decision as follows:

Declare population  $\pi_i$  as a non-best population and exclude it from further experiment if there exists some population  $\pi_h$  such that

$$N_h(t) < m \quad \text{and} \quad E[\theta_h | y_h(t), m] \geq E[\theta_i | y_i(t, T), N_i(t)] \quad (4.2a)$$

or

$$N_h(t) = m \quad \text{and} \quad E[\theta_h | y_h(t), m] > E[\theta_i | y_i(t, T), N_i(t)] \quad (4.2b)$$

where

$$y_h(t) = \sum_{j=1}^{N_h(t)} Q(y_{hj}) + (m - N_h(t))Q(t) \quad (4.3a)$$

and

$$y_i(t, T) = \sum_{j=1}^{N_i(t)} Q(y_{ij}) + (m - N_i(t))Q(T). \quad (4.3b)$$

Let  $S(t)$  denote the indices of the contending populations for the best at time  $t$ . That is,

$$S(t) = \{i : N_h(t) < (=)m \text{ and } E[\theta_i|y_i(t, T), N_i(t)] > (\geq)E[\theta_h|y_h(t), m], h \neq i\}. \quad (4.4)$$

The following lemma shows that for any  $t$ ,  $0 < t < T$ ,  $S(t)$  is not empty.

**Lemma 4.2** *For any  $0 < t < T$ , the set  $S(t)$  defined by (4.4) is not empty.*

**Proof.** Let

$$S'(t) = \{i : E[\theta_i|y_i(t), N_i(t)] = \max_{1 \leq h \leq k} E[\theta_h|y_h(t), N_h(t)]\}.$$

Then  $S'(t)$  is not empty. We prove that  $S'(t)$  is a subset of  $S(t)$ . For  $i \in S'(t)$  and any  $h \neq i$ ,  
if  $N_h(t) < m$ , then

$$E[\theta_i|y_i(t, T), N_i(t)] \geq E[\theta_i|y_i(t), N_i(t)] \geq E[\theta_h|y_h(t), N_h(t)] > E[\theta_h|y_h(t), m];$$

if  $N_h(t) = m$ , then

$$E[\theta_i|y_i(t, T), N_i(t)] \geq E[\theta_i|y_i(t), N_i(t)] \geq E[\theta_h|y_h(t), N_h(t)] \geq E[\theta_h|y_h(t), m].$$

In either situation, we see that  $i \in S(t)$ . Hence  $S'(t) \subset S(t)$ .

Now, the experiment terminates as soon as there is a time  $t$ ,  $0 < t < T$ , such that  $|S(t)| = 1$  and in this case, we select the population with its index in  $S(t)$  as the best population. Otherwise, the experiment goes on until time  $T$ . Let

$$S(T) = \{i : E[\theta_i|y_i, N_i] = \max_{j \in S(T^-)} E[\theta_j|y_j, N_j]\}, \quad (4.5)$$

where  $S(T^-)$ , which is not empty by Lemma 4.2, denotes the set of the indices of those populations having not been eliminated before time  $T$ . Then, a uniformly randomized selection is made from  $S(T)$ .

From the above description, we see that the early selection rule can possibly make a final selection earlier than the termination time  $T$ . Denote this early selection rule by  $\tilde{\delta}_G^* = (\delta_{G1}^*, \dots, \delta_{Gk}^*)$ . Then, we have the following theorem.

**Theorem 4.3** Under the loss function  $L(\theta)$ ,  $\tilde{\delta}_{Gi}^* = \tilde{\delta}_{Gi}(\tilde{y}, \tilde{n})$  for all  $1 \leq i \leq k$ ,  $\tilde{y} \in \mathcal{Y}_n$  and  $\tilde{n} \in \mathcal{N}$ , where  $\tilde{\delta}_{Gi}(\tilde{y}, \tilde{n})$  is defined by (4.1) and (2.7).

Let  $t_1 = \inf\{t : |S(t)| = 1, 0 < t \leq T\} \wedge T$ , where  $a \wedge b = \min(a, b)$ . Then Theorem 4.3 is equivalent to the following theorem.

**Theorem 4.4**  $S(t_1) = A(\tilde{y}, \tilde{n})$  for all  $(\tilde{y}, \tilde{n})$ .

**Proof.** Case 1. If  $t_1 < T$ , then  $|S(t_1)| = 1$ . Without loss of generality, we let  $\pi_k$  be the population with index in the set  $S(t_1)$ . Since  $A(\tilde{y}, \tilde{n})$  contains at least one element, it suffices to show that  $i \notin A(\tilde{y}, \tilde{n})$  for all  $i \neq k$ . Since  $i \notin S(t_1)$ , it means that population  $\pi_i$  is eliminated at some prior time, say  $t_0$ . That is, at time  $t_0$ , for some  $\pi_h$ , either

$$N_h(t_0) < m \quad \text{and} \quad E[\theta_h|y_h(t_0), m] \geq E[\theta_i|y_i(t_0, T), N_i(t_0)] \quad (4.6a)$$

or

$$N_h(t_0) = m \quad \text{and} \quad E[\theta_h|y_h(t_0), m] > E[\theta_i|y_i(t_0, T), N_i(t_0)]. \quad (4.6b)$$

Now, note that  $N_i(t)$  is a nondecreasing function of  $t \in (0, T]$  and  $N_i(t) \leq m$ . Also, by (4.3a) and (4.3b),  $y_h(t)$  is nondecreasing in  $t$  and  $y_i(t, T)$  is nonincreasing in  $t$ . Especially, we have

$$N_h = H_h(T) \leq m, \quad N_i(t) \leq N_i(T) = N_i, \quad y_i(t_0, T) \geq y_i$$

and

$$y_h = \begin{cases} > y_h(t_0) & \text{if } N_h(t_0) < m, \\ = y_h(t_0) & \text{if } N_h(t_0) = m. \end{cases}$$

Thus, when  $N_h(t_0) = m$ , then  $N_h = m$ . Then by Lemma 4.1 and (4.6b),

$$\begin{aligned} E[\theta_h|y_h, N_h] &= E[\theta_h|y_h(t_0), m] \\ &> E[\theta_i|y_i(t_0, T), N_i(t_0)] \\ &\geq E[\theta_i|y_i, N_i]. \end{aligned}$$

When  $N_h(t_0) < m$ , then  $y_h > y_h(t_0)$  and  $N_h \leq m$ . Therefore, by Lemma 4.1 and (4.6a),

$$\begin{aligned} E[\theta_h|y_h, N_h] &= E[\theta_h|y_h(t_0), m] \\ &> E[\theta_i|y_i(t_0, T), N_i(t_0)] \\ &\geq E[\theta_i|y_i, N_i]. \end{aligned}$$

In either situation, we see that  $i \notin A(\tilde{y}, n)$ .

Case 2. If  $t_1 = T$ , we need to prove that

(a)  $i \notin S(T)$  implies  $i \notin A(\tilde{y}, n)$ , and (b)  $i \in S(T)$  implies  $i \in A(\tilde{y}, n)$ .

We prove (a) first. Suppose  $i \notin S(T)$ . Then,  $\pi_i$  is eliminated at a time  $t_0 \leq T$  by some other  $\pi_h$ .

If  $t_0 \leq T$ , this reduces to the situation discussed in Case 1.

If  $t_0 = T$ , then by (4.5),  $E[\theta_h|y_h, N_h] > E[\theta_i|y_i, N_i]$ . Therefore, by the definition of  $A(\tilde{y}, \tilde{n})$ ,  $i \notin A(\tilde{y}, \tilde{n})$ .

For (b), we have firstly  $A(\tilde{y}, \tilde{n}) \subset S(T) \subset S(T^-)$  by (a) and definition of  $S(T)$  and  $S(T^-)$ . If  $i \in S(T^-)$ ,

$$\begin{aligned} E[\theta_i|y_i, N_i] &= \max_{j \in S(T^-)} E[\theta_j|y_j, N_j] \\ &\geq \max_{j \in S(T)} E[\theta_j|y_j, N_j] \\ &\geq \max_{j \in A(\tilde{y}, \tilde{n})} E[\theta_j|y_j, N_j]. \end{aligned}$$

This means  $i \in A(\tilde{y}, \tilde{n})$ . The proof now is completed.

## § 5 An Example

We use the simulated data to illustrate how the early selection rule works. Suppose that we have five populations  $\pi_i$ ,  $i = 1, 2, 3, 4, 5$ . The lifetime of the population  $\pi_i$  follows a Weibull distribution with density

$$f(x|\theta_i) = \frac{2x}{\theta_i^2} \exp[-(\frac{x}{\theta_i})^2], \quad x > 0.$$

The unknown parameters  $\theta_1, \dots, \theta_5$  are simulated independently from  $U(0, 1)$ . That is,  $\theta_1, \dots, \theta_5$  are independent and identically distributed with  $U(0, 1)$ . Ten observations are simulated independently from each population. The data are listed in the following table.

$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$
0.10	0.06	0.57	0.80	0.67
0.33	0.09	0.56	1.19	0.90
0.50	0.05	0.61	0.70	1.02
0.17	0.04	0.75	0.36	0.64
0.50	0.02	0.68	0.88	0.74
0.13	0.02	0.22	0.52	0.90
0.13	0.07	0.47	0.71	0.94
0.28	0.10	0.45	0.77	0.86
0.36	0.07	0.69	0.52	0.59
0.10	0.07	1.01	0.24	0.90

We want to select a population with the largest mean lifetime. Since the mean lifetime of the population  $\pi_i$  is proportional to  $\theta_i$ , what we need to do is to find a population with the parameter  $\theta_{[k]}$ . Suppose that the type-I censoring scheme is planned before the life-testing experiment and the censoring time is set to be  $T = 1$ . Therefore we obtain the following table.

	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$
$E[\theta_i y_i, n_i]$	0.14	0.03	0.31	0.34	0.83

According to the selection rule  $\tilde{\delta}_G$ , at the end of the experiment, we select  $\pi_5$  as the best population.

However, if the early selection rule  $\tilde{\delta}_G^*$  is applied, we can make the selection before  $T = 1$  and end the experiment earlier. According to the selection rule  $\tilde{\delta}_G^*$ , at time  $t$ ,  $0 < t < T = 1$ , exclude the population  $\pi_i$  as a non-best population and remove it from further experiment if there exists some population  $\pi_h$  such that

$$N_h(t) < m \quad \text{and} \quad E[\theta_h|y_h(t), m] \geq E[\theta_i|y_i(t, T), N_i(t)]$$

or

$$N_h(t) = m \quad \text{and} \quad E[\theta_h|y_h(t), m] > E[\theta_i|y_i(t, T), N_i(t)].$$

According to this rule, at  $t_1 = 0.88$ , all the populations  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$  and  $\pi_4$  are removed from the experiment and the population  $\pi_5$  is selected as the best. So the experiment can be ended at  $t_1 = 0.88$  and the time saved is 0.12 or 12%.

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